ELSEVIER

Acta Psychologica 100 (1998) 85–96

acta psychologica

The 2/3 power law: When and why?

Réjean Plamondon *, Wacef Guerfali [1]

Laboratoire Scribens, École Polytechnique de Montréal, Département de génie électrique et de génie informatique, C.P. 6079, Succursale Centre-Ville, Montréal, Québec, Canada H3C 3A7

Received 20 November 1997; received in revised form 8 April 1998; accepted 7 June 1998

Abstract

This paper presents the origin of some reported observations, which links the kinematics of handwriting with a movement trajectory, best known as the 2/3 power law. Using computer simulations, it is shown that the vectorial delta-lognormal model recently proposed to describe 2D movements can successfully simulate these phenomena. Although the power law has been found to be a good predictor in many experimental conditions, a few experiments have shown that the law does not apply to all graphic movements. Using the vectorial delta-lognormal model, the conditions under which a 2/3 power relationship can be observed are presented and the reasons why it does not seem to be verified for more general handwritten patterns are highlighted. © 1998 Published by Elsevier Science B.V. All rights reserved.

PsycINFO classification: 2330

Keywords: 2/3 Power law; Vectorial delta-lognormal model; Handwriting; Motor-perception interaction; Movement trajectory; Delta-lognormal law

1. Introduction

In the early eighties, the correlation that exists between the kinematics of handwriting and the movement trajectory was formalized in terms of an equation, known

* Corresponding author. Tel.: (514) 340-4711 (4539); fax: (514) 340-4600; e-mail: rejean.plamondon@mail.polymtl.ca

[1] E-mail: guerfali@scribens.polymtl.ca.

PII: S 0 0 0 1 - 6 9 1 8 (9 8) 0 0 0 2 7 - 4

as the 2/3 power law. This law links the curvature $c(t)$ of a trajectory to the angular velocity $v_\theta(t)$ of the pentip that generates it by a 2/3 power (Lacquaniti et al., 1983, 1984).

$$v_\theta(t) = kc^{2/3}(t). \tag{1}$$

Taking into account the intrinsic relationship that exists between the curvilinear velocity $v(t)$ and the angular velocity $v_\theta(t)$ where ($v(t) = R\, v_\theta(t) = v_\theta(t)/C$), the same expression can also be written in terms of a relationship between the curvilinear velocity $v(t)$ and the radius of curvature of the trajectory $r(t)$

$$v(t) = k\, r^{1/3}(t). \tag{2}$$

Since then, the law has been shown to be valid for a certain class of movements and it has been slightly modified and adapted over the years to cover an even larger set of movements. The most recent formulation (Viviani and Schneider, 1991; Viviani and Stucchi, 1992) can be expressed by Eq. (3).

$$v(t) = k(t)\left(\frac{r(t)}{1 + \alpha r(t)}\right)^{\beta}, \tag{3}$$

where $k(t)$ is a positive function of time, α is a positive constant and the constant β is approximately equal to $\frac{1}{3}$. It should be noted that if $k(t)$ is a constant and $\alpha = 0$, then Eq. (3) is equivalent to Eq. (2).

Although the power law has been found to be a good predictor in many experimental conditions (Viviani, 1986; Viviani and Cenzato, 1985), a few experiments have shown that the law does not apply to many graphic movements (Thomassen and Teuilings, 1985; Wann et al., 1988).

This paper proposes an explanation for the emergence of a 2/3 power law and it highlights, using computer simulations, the conditions under which this relationship can be observed. The rationale is based on the delta-lognormal law (Plamondon, 1993a,b, 1995a,b) which describes the velocity profile of a single stroke, and on strategies for the superimposition of strokes in complex movements (Plamondon and Guerfali, 1998).

The first part of the paper briefly describes the general handwriting generation model used for computer simulations, pointing out its origin and its main properties. The second part of the paper shows the observations and simulations that explain under which conditions a 2/3 power law is observed.

2. Handwriting generation model

According to the kinematic theory (Plamondon, 1993a,b, 1995a,b), simple human movements can be described in the velocity domain as the response of the synergistic action of an agonist and an antagonist neuromuscular network. Each network reacts to an input command $D_1 U_0(t - t_0)$ (for the agonist) or $D_2 U_0(t - t_0)$ (for the antagonist) with an impulse response that can be described by a lognormal function

(Plamondon, 1993a, 1995a). Each lognormal impulse response $\Lambda(t\,;\ t_0, \mu_i, \sigma_i^2)$ can be characterized by three parameters: the activation time t_0, the parameter μ_i which reflects its logtime delay, and σ_i^2 which reflects the logresponse time of the network (Plamondon, 1993a, 1995a). The output of the agonist (or the antagonist) system is thus the convolution of $D_1 U_0(t - t_0)$ (or $D_2 U_0(t - t_0)$) with a lognormal impulse response, and the resulting curvilinear velocity $v(t)$ of a single movement is then described by subtracting the weighted impulse response of the antagonist network from the agonist one;

$$v(t) = D_1 \Lambda(t\,;\ t_0, \mu_1, \sigma_1^2) - D_2 \Lambda(t\,;\ t_0, \mu_2, \sigma_2^2), \tag{4}$$

where

$$\Lambda(t\,;\ t_0, \mu_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi}\sigma_i(t - t_0)} \exp^{-\frac{1}{2\sigma_i^2}(\ln(t-t_0)-\mu_i)^2}. \tag{5}$$

Eq. (4) is referred to as a delta-lognormal law. This law is the basis for the vectorial delta-lognormal model, a movement generation model that describes individual two-dimensional movements (Plamondon, 1995c). According to this model, a neuromuscular synergy controls the velocity vector, the magnitude of which follows a delta-lognormal law. Apart from the seven parameters of the delta-lognormal function $(t_0, D_1, \mu_1, \sigma_1^2, D_2, \mu_2, \sigma_2^2)$, each velocity vector is also characterized in the space domain by three postural parameters that globally reflect the geometric properties of the set of muscles and joints used in a particular movement: the starting point P_0, the starting direction θ_0 and the global curvature C_0 of the stroke (Plamondon and Guerfali, 1998). The curvature is considered positive if the movement is clockwise, and negative otherwise.

The angular direction of the velocity vector for a single stroke can be deduced from the intrinsic relation that links the angular and the curvilinear velocities (Plamondon, 1995c; Guerfali and Plamondon, 1995a) (Eq. (6)).

$$\theta\,(t) = \theta_0 + C_0 \int_{t_0}^{t} \left|\vec{v}(\tau)\right| \mathrm{d}\tau. \tag{6}$$

A single movement, hereafter called a stroke, can be represented in the space and velocity domains by a velocity vector starting at time t_0 at point P_0 with an initial direction θ_0, and moving along a circular path of length $D_1 - D_2$ with a constant curvature C_0. According to the kinematic theory, the movement described by this model will reach its target with a movement time that is related to the ratio of the agonist and antagonist commands, D_1/D_2 (Plamondon, 1993b, 1995b).

As its name suggests, the vectorial delta-lognormal model considers the kinematics of a single movement in terms of a vector moving along a circular path with a delta-lognormal velocity profile. Complex movements are then the result of the time overlap of two or more velocity vectors. The resulting curvilinear velocity vector can be characterized by its instantaneous magnitude (Eq. (7)), which is the magnitude of the sum of individual velocity vectors, and its instantaneous orientation as described by Eq. (8).

$$v(t) = \left|\vec{v}(t)\right| = \left|\sum_{i=1}^{n} \vec{v}_{(i)}(t - t_{0(i)})\right|, \tag{7}$$

$$\theta(t) = \arctan\left(\frac{\sum_{i=1}^{n}\left|\vec{v}_{(i)}(t - t_{0(i)})\right| \sin(\theta_{(i)}(t - t_{0(i)}))}{\sum_{i=1}^{n}\left|\vec{v}_{(i)}(t - t_{0(i)})\right| \cos(\theta_{(i)}(t - t_{0(i)}))}\right), \tag{8}$$

where $t_{0(i)}$ is the time occurrence of the i^{th} stroke.

The magnitude of the angular velocity is obtained from the time derivative of (8) as given in Appendix A. Using this expression (A.2) one can explain the origin and the shape of the angular velocity as well as the phase shift that exists between the curvilinear and the angular velocity in complex movements. One can also show how the angular velocity signal emerges from the vectorial summation process without any specific and independent control (Guerfali and Plamondon, 1995b; Plamondon 1995c; Guerfali, 1996).

3. Prediction of the 2/3 power law

With the use of the vectorial delta-lognormal model and computer simulations, one can predict under what conditions a 2/3 power law is observed between the angular velocity and the trajectory curvature (an analytical demonstration is also given in Plamondon and Guerfali (1998)). First, it is self-evident that, for a simple movement (a single stroke) where the curvature is almost constant $c(t) = C_0$, no power law will be observed (Plamondon and Guerfali, 1997; 1998). The intrinsic relationship that exists between angular and curvilinear velocity expressed in Eq. (9) shows this impossibility.

$$v_\theta(t) = C_0 v(t) \neq k C_0^{2/3}. \tag{9}$$

Indeed, under these conditions $v_\theta(t)$ is proportional to $v(t)$ and thus also obey a delta-lognormal law.

The computer simulation, presented in Fig. 1, shows a trajectory made up of two strokes superimposed in time, which is the simplest condition in which to observe a 2/3 power law. Each stroke (i) is characterized by a curvature C_0, an initial angular direction θ_0 and the magnitude of its velocity vector $|\vec{v}(t - t_0)|$, as described by a delta-lognormal law. Assuming for simplicity that the first stroke starts at $t_{0(1)} = 0$ and the second at time $t_{0(2)}$, the velocity profile of the composed movement is described by Eq. (10).

$$\vec{v}(t) = \vec{v}_{(1)}(t) + \vec{v}_{(2)}\left(t - t_{0(2)}\right). \tag{10}$$

The analytical development of Eq. (10) as detailed in Plamondon and Guerfali (1998) highlights some specific conditions under which a 2/3 power law can be observed. Using computer simulations, we have tried to study and generalize this approach here. A first set of conditions deals with the difference between the angular

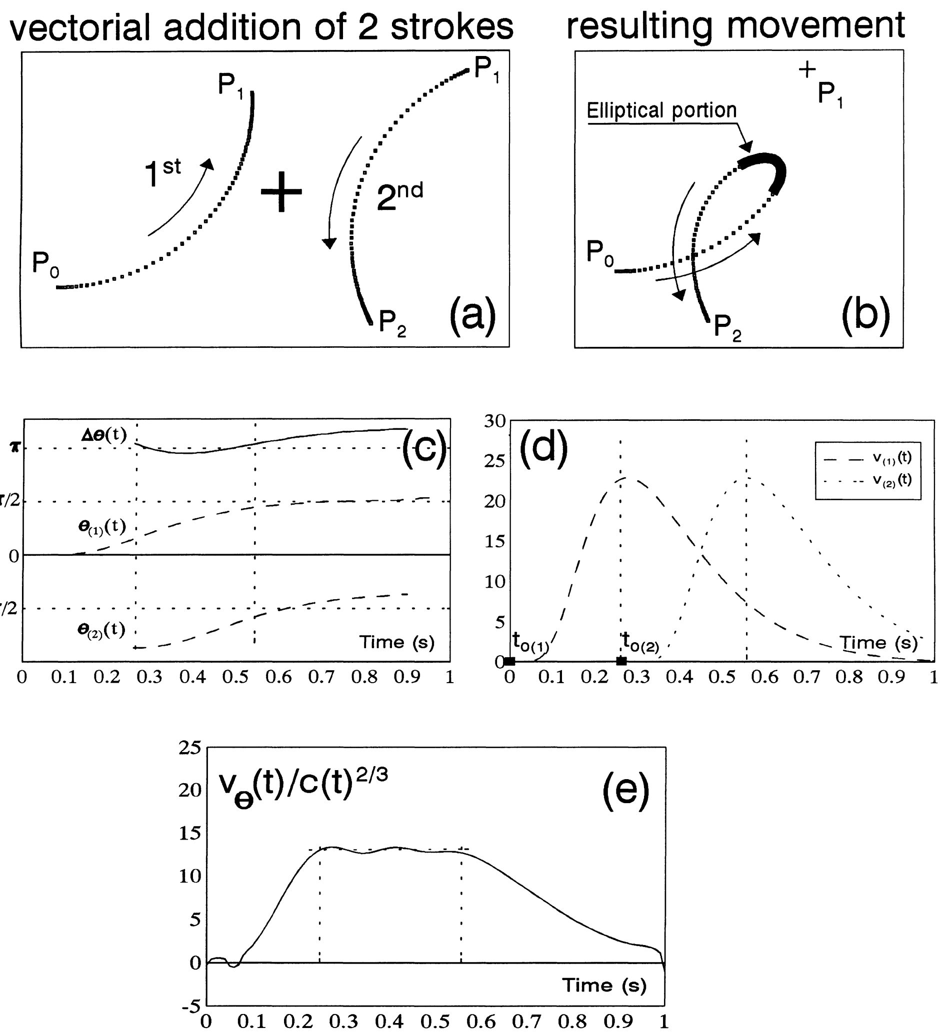

Fig. 1. A movement made up of two strokes. (a) The parameters of the basic strokes are: $D_{1(1)} = 10.0$; $D_{2(1)} = 2.0$; $t_{0(1)} = 0.0$; $\mu_{1(1)} = -1.0$; $\mu_{2(1)} = -0.5$; $\sigma_{1(1)} = \sigma_{2(1)} = 0.5$; $C_{0(1)} = 0.2$; $\theta_{0(1)} = 0.0$; $D_{1(2)} = 10.0$; $D_{2(2)} = 2.0$; $t_{0(2)} = 0.28$; $\mu_{1(2)} = -1.0$; $\mu_{2(2)} = -0.5$; $\sigma_{1(2)} = \sigma_{2(2)} = 0.5$; $C_{0(2)} = 0.2$; $\theta_{0(2)} = -2.73$. (b) The resulting two-stroke movement. (c) The angular variation $\Delta\theta(t)$. (d) The curvilinear velocities of the strokes. (e) The ratio $v_\theta(t)/c^{2/3}(t)$.

direction $\Delta\theta(t)$ of the two consecutive strokes. This difference must be preferably a multiple of $\pi/4$. In our example in Fig. 1(c), $\Delta\theta(t)$ is almost equal to π for a few tenths of a second. The second set of conditions deals with the time occurrence $t_{0(2)}$ of the second stroke. This stroke must start before the end of the first stroke in such a way that the delay between the velocity signals of the two strokes will compensate for their angular phase shift $\Delta\theta(t)$ as well as keeping the sum of the two speeds almost constant. In other words, the increase in the velocity $v_{(2)}(t - t_{0(2)})$ of the second stroke must compensate for the decrease in the velocity of the first stroke (see Fig. 1(d)). For the time interval when these conditions are met, the resulting portion of the trajectory can be approximated by a hyperbola or an ellipse, and the angular velocity is linked to the curvature by a 2/3 power law. The ratio of $v_\theta(t)/c(t)^{2/3}$ is thus constant for that period, as depicted in Fig. 1(e). For the two-stroke movements illustrated in Fig. 1, the elliptical portion of the resulting trajectory (Fig. 1(b)) is identified by a bold line, corresponding to the almost constant part of the plotted ratio $v_\theta(t)/c^{2/3}(t)$ in Fig. 1(e).

The 2/3 power relationship will thus be observed only for some particular conditions where the values of the velocity vectors, C_0, θ_0 and $t_{0(2)}$ ensure a large zone where the previous conditions on $\Delta\theta(t)$ and $t_{0(2)}$ between two successive strokes will be met. A change in any of these parameters will affect the resulting trajectory and deteriorate the 2/3 power approximation. Fig. 2 shows, for example, the effect of a variation in the timing parameter $t_{0(2)}$ on the previous trajectory in Fig. 1(b), and on the ratio $v_\theta(t)/c^{2/3}(t)$, all other parameters being left constant. As one can see, a slight departure of a few hundredths of a second in the time delay between two strokes drastically reduces the domain of validity of the 2/3 power law.

The 2/3 power law becomes more apparent when the same pattern of circular strokes is repeated, while superimposing them to generate a sequence of loops. Fig. 3 shows a simulation of a superimposition of three loops (8 strokes). Here, the

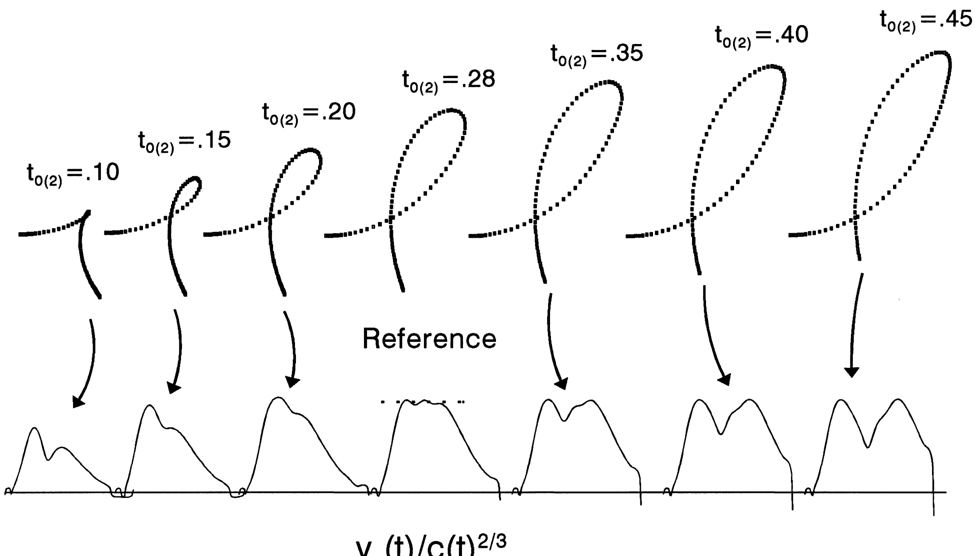

Fig. 2. Effect of the variation of the time occurrence of the second stroke $t_{0(2)}$ on the resulting trajectory and on the ratio $v_\theta(t)/c^{2/3}(t)$ for the two-stroke simulated movement presented in Fig. 1.

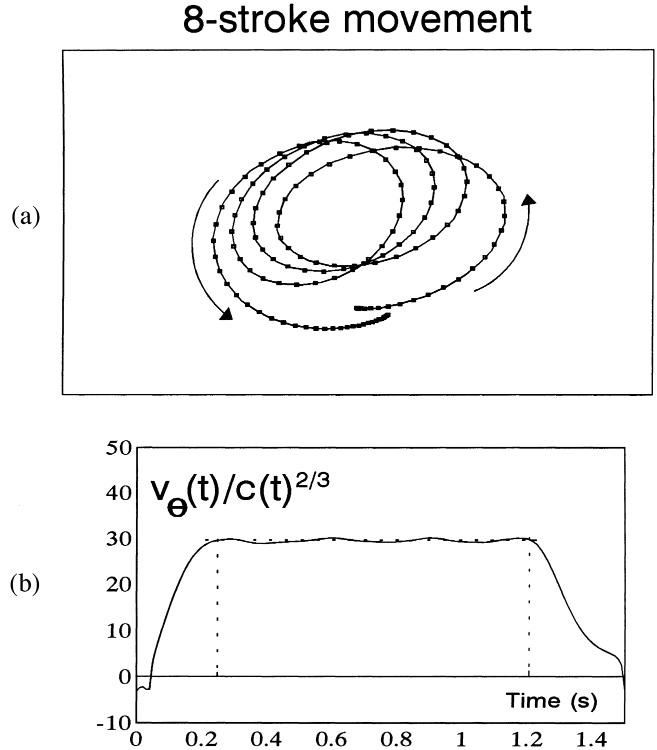

Fig. 3. A three-loop (8 strokes) movement: (a) the resulting simulated movement; (b) the ratio $v_\theta(t)\,/\,c^{2/3}(t)$ as a function of time.

resulting elliptical trajectory presents a larger part where the 2/3 power law applies, which is consistent with the study of Lacquaniti et al., (1983, 1984).

When the ratio $v_\theta(t)/c^{2/3}(t)$ is plotted as a function of time for real words written by a human subject, roughly elliptical portions can also be identified in the trajectory (see Fig. 4(a) and (b) , elliptical portions (e_1), (e_2), (e_3) and (e_4)). For those parts where the 2/3 power law seems to be valid, it is verified, by means of analysis-by-synthesis (Guerfali and Plamondon, 1998), that the previous conditions regarding $\Delta\theta(t)$ and $\left(t_{0(2)} - t_{0(1)}\right)$ roughly apply. Fig. 4(c) shows the strokes extracted by means of analysis-by-synthesis, illustrating one possible action plan that has been used by the subject to generate the whole movement (Plamondon and Guerfali, 1998; Guerfali and Plamondon, 1998). Here, the first stroke starts at point P_0 (top of the letter "s"), while subsequent strokes will target successive virtual points (numbered from 1 to 11), which they will not usually hit because of superimposition phenomena (except in cases where there are no strokes overlapping and velocity tends toward zero). The bold parts of the strokes, in Fig. 4(c), correspond to the superimposed regions where elliptical portions (e_1), (e_2), (e_3) and (e_4) are observed and where the two sets of conditions mentioned in Section 3 are met. As can be seen, $\Delta\theta(t)$ is almost

(a)

(b)

(c)

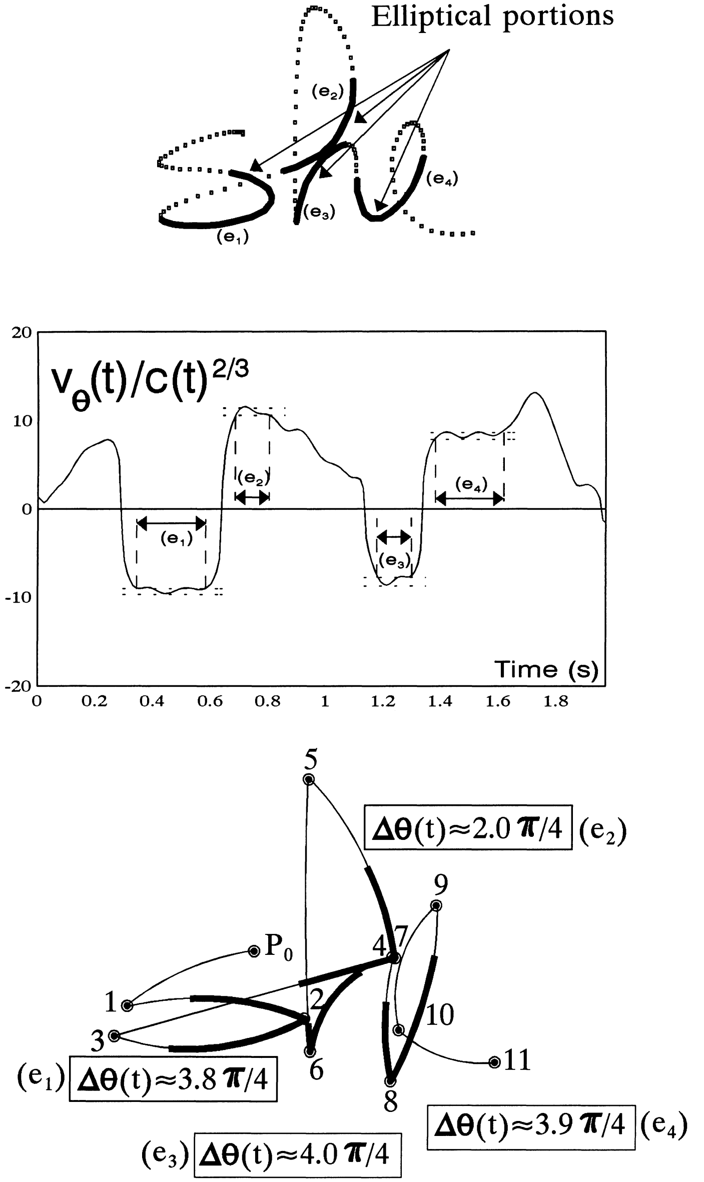

Fig. 4. Real handwriting produced by a human subject and analyzed with the Vectorial Delta-lognormal Model: (a) original word with highlights on the elliptical portions; (b) the ratio $v_\theta(t)\,/\,c^{2/3}(t)$ as a function of time; (c) segmented overlapping strokes as extracted from analysis-by-synthesis and regions of superimposition where conditions apply.

constant and almost a multiple of $\pi/4$. Comparing several simulations, it is shown that for real handwriting the emergence of the 2/3 power law is less clear than for simple oscillatory movements where the law is more apparent.

Even for the more general relationship (Eq. (3)), where α and β can be adjusted around their initial values ($\alpha = 0.0$ and $\beta = 1/3$), as shown respectively in Figs. 5(a) and (b), $k(t)$(see Eq. (3)) is not constant for real handwriting, except as regards a few small portions of the trajectory. In the particular case illustrated in Fig. 5, the

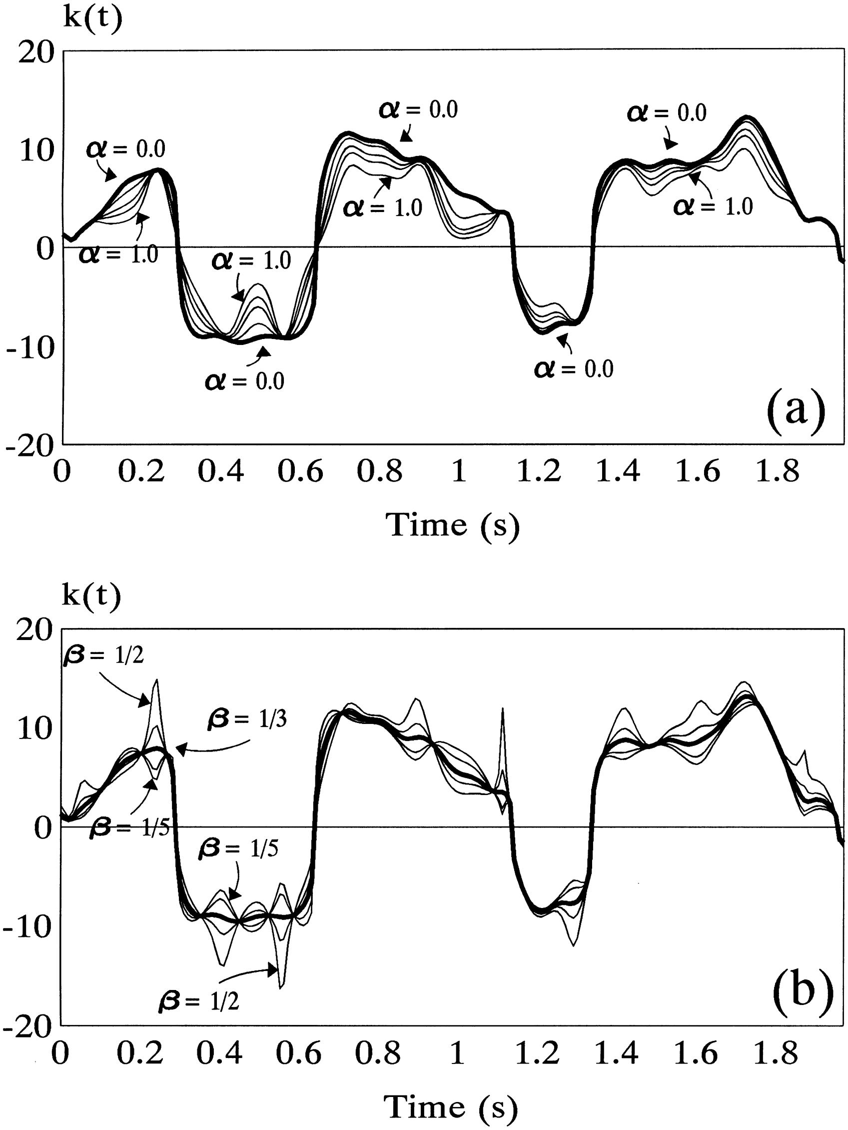

Fig. 5. Evolution of $k(t)$(Eq. (3)) for different values of α and β: (a) $\alpha = (0.0, 0.1, 0.3, 0.5, 1.0)$; $\beta = 1/3$ (b) $\alpha = 0.0$; $\beta = (1/2, 1/2.5, 1/3, 1/4, 1/5)$ (for the word *she* presented in Fig. 4).

original values ($\alpha = 0.0$ and $\beta = 1/3$) even constitute the best conditions for observing a power law in some parts of the trajectory, and the generalization proposed by Eq. (3) does not constitute an improvement.

From a neuromuscular point of view, the observation of a 2/3 power law in a specific trajectory results from the vectorial superimposition of at least two individual movement units where both spatial (or angular) constraints and temporal (or sequencing) constraints have to be met simultaneously. These constraints can be represented at the action plan level (see Fig. 4(c)) in terms of a sequence of virtual targets linked by discontinuous circular strokes, each one having its own specific activation time $(t_{0(i)})$. For two consecutive strokes having an angular phase shift of $n\pi/4$ and specific time shift of $t_{0(i)} - t_{0(i-1)}$, a 2/3 power relationship will emerge in the trajectory, upon execution of the action plan through a delta-lognormal neuromuscular system.

4. Conclusion

Using the vectorial delta-lognormal model (Plamondon and Guerfali, 1998), we have shown in this paper under what conditions a 2/3 power law can be observed in a multiple-stroke trajectory. First, successive strokes must be out of phase by multiple values of $\pi/4$ for a certain period; second, the time shift between successive strokes must compensate for that phase shift and be such that the increase in the velocity of the second stroke compensates for the decrease in the velocity of the first stroke during that same period. When these conditions are met, the trajectory becomes elliptical or hyperbolic and a 2/3 power law is observed for that period.

Computer simulations show that elliptical portions of trajectories will occur more frequently for simple patterns than for more complex patterns, which involve multiple strokes and discontinuity points. The controversial results that have been reported about the 2/3 power law are reproduced by our simulation of the vectorial delta-lognormal model that shows in which cases the law seems to apply. It seems clear now, both by simulation and by analysis of real handwriting movements, that for non-oscillatory movements the 2/3 power law does not hold for the major parts of a trajectory. It can be observed, however, that for some roughly elliptical part of handwriting the law can be valid. It is also concluded that the more general formulation of the power law (Eq. (3)) does not substantially help in the analysis of natural cursive handwriting movements.

In a more general perspective, this paper opens a new window through which to look at handwriting (Plamondon, 1998). The vectorial delta-lognormal model provides a framework for analyzing, more specifically, those portions of the trajectory where the 2/3 power law seems to apply. The constituent strokes can be extracted and characterized (Guerfali and Plamondon, 1998) in terms of their nine parameters. These parameters can be analyzed statistically so that, eventually, new insights into the motor-perceptual interaction possibly involved in this phenomenon will be provided.

Appendix A

The angular velocity can be obtained from the time derivative of Eq. (8).

$$\theta(t) = \arctan\left(\frac{\sum_{i=1}^{n}\left|\vec{v}_{(i)}\left(t-t_{0(i)}\right)\right| \sin\left(\theta_{(i)}\left(t-t_{0(i)}\right)\right)}{\sum_{i=1}^{n}\left|\vec{v}_{(i)}\left(t-t_{0(i)}\right)\right| \cos\left(\theta_{(i)}\left(t-t_{0(i)}\right)\right)}\right) = \arctan\left(\frac{f(t)}{g(t)}\right) \quad (A.1)$$

The analytical formula for the angular velocity can then be expressed as

$$v_{\theta}(t) = \frac{(\mathrm{d}f(t)/\mathrm{d}t)\, g(t) - f(t)\, (\mathrm{d}g(t)/\mathrm{d}t)}{f^2(t) + g^2(t)}, \quad (A.2)$$

where

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \sum_{i=1}^{n}\left(\frac{\mathrm{d}\left|\vec{v}_{(i)}\left(t-t_{0(i)}\right)\right|}{\mathrm{d}t} \sin\left(\theta_{(i)}\left(t-t_{0(i)}\right)\right) + C_{0(i)}\left|\vec{v}_{(i)}\left(t-t_{0(i)}\right)\right|^2 \cos\left(\theta_{(i)}\left(t-t_{0(i)}\right)\right)\right), \quad (A.3)$$

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \sum_{i=1}^{n}\left(\frac{\mathrm{d}\left|\vec{v}_{(i)}\left(t-t_{0(i)}\right)\right|}{\mathrm{d}t} \cos\left(\theta_{(i)}\left(t-t_{0(i)}\right)\right) + C_{0(i)}\left|\vec{v}_{(i)}\left(t-t_{0(i)}\right)\right|^2 \sin\left(\theta_{(i)}\left(t-t_{0(i)}\right)\right)\right). \quad (A.4)$$

References

Guerfali, W., 1996. Modèle delta lognormal vectoriel pour l'analyse du mouvement et la génération de l'écriture manuscrite. Ph.D. dissertation, École Polytechnique de Montréal.

Guerfali, W., Plamondon, R., 1995a. The delta lognormal theory for the generation and modelling of cursive characters. Proceedings of the Third International Conference on Document Analysis and Recognition. pp. 495–498.

Guerfali, W., Plamondon, R., 1995b. Control strategies for handwriting generation. Proceedings of the Seventh Biennial Conference of the International Graphonomics Society. London, Ontario, Canada, pp. 64–65.

Guerfali, W., Plamondon, R., 1998. A new method for the analysis of simple and complex planar rapid movements. Journal of Neuroscience Methods (in press).

Lacquaniti, F., Terzuolo, C., Viviani, P., 1983. The law relating the kinematic and figural aspects of drawing movements.. Acta Psychologica 54 (3), 115–130.

Lacquaniti, F., Terzuolo, C., Viviani, P., 1984. Global metric properties and preparatory processes in drawing movements. In: Kornblum, S., Requin, J. (Eds.), Preparatory States and Processes. Hillsdale, pp. 357–370.

Plamondon, R., 1993a. The generation of rapid human movements: Part I: A Δlog-normal Law. EPM/RT-93/4.

Plamondon, R., 1993b. The generation of rapid human movements: Part II: Quadratic and Power Laws. EPM/RT-93/5.

Plamondon, R., 1995a. A kinematic theory of rapid human movements: Part I: Movement representation and generation. Biological Cybernetics 72 (4), 295–307.

Plamondon, R., 1995b. A kinematic theory of rapid human movements: Part II Movement time and control. Biological Cybernetics 72 (4), 309–320.

Plamondon, R., 1995c. A delta–lognormal model for handwriting generation. Proceedings of the Seventh Biennial Conference of the International Graphonomics Society. London, Ontario, Canada, pp. 126–127.

Plamondon, R., 1998. The kinematic theory: A new window to study and analyze simple and complex human movements. Behavioral and Brain Sciences 20 (2), 325–348.

Plamondon, R., Guerfali, W., 1997. The Origin of the 2/3 Power Law. Proceedings of the Eight Biennial Conference of the International Graphonomics Society. Genova, Italy, pp. 17–18.

Plamondon, R., Guerfali, W., 1998. The generation of handwriting with delta–lognormal synergies. Biological Cybernetics 78 (2), 119–132.

Thomassen, A.J.W.M., Teulings, H.L., 1985. Time, size and shape in handwriting: Exploring spatio–temporal relationships at different levels. In: Michon, J.A., Jackson, J.L. (Eds.), Time, Mind, and Behavior. Springer, Berlin, pp. 253–263.

Viviani, P., Cenzato, M., 1985. Segmentation and coupling in complex movements. Journal of Experimental Psychology: Human Perception and Performance 11, 828–845.

Viviani, P., 1986. Do units of motor action really exist? In: Heuer, H., Fromm, C. (Eds.), Generation and Modulation of Action Patterns. Springer, Berlin, pp 201–216..

Viviani, P., Schneider, R., 1991. A development study of the relation between geometry and kinematics in drawing movements. Journal of Experimental Psychology: Human Perception and Performance 17, 198–218.

Viviani, P., Stucchi, N., 1992. Biological movements look constant: Evidence of motor perceptual interactions. Journal of Experimental Psychology: Human Perception and Performance 18, 603–623.

Wann, J., Nimmo-Smith, I., Wing, A.M., 1988. Relation between velocity and curvature in movement: Equivalence and divergence between a power law and a minimum jerk model. Journal of Experimental Psychology: Human Perception and Performance 14 (4), 622–637.